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# Expected Utility, $\mu$–$\sigma$ Preferences, and Linear Distribution Classes: A Further Result

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*Abstract*

This article is an extension of Meyer and Sinn's results on the representation of arbitrary von Neumann-Morgenstern functions in $\mu$-$\sigma$ space when the probability distributions to be compared belong to a linear distribution class. It shows that, when absolute risk aversion decreases, stays constant, or increases not too fast, an increase in $\sigma$, given $\mu$, increases the indifference curve slope: increased riskiness increases the required marginal compensation for risk when risk is measured by the standard deviation of wealth or income.

## 1. Linear distribution classes

It is frequently believed that the $\mu$–$\sigma$ criterion is compatible with expected utility maximization only if the utility function is quadratic or if there are normal distributions. Recent contributions by Meyer (1987, 1989) and Sinn (1983, pp. 56-57, 115-120; 1989) have emphasized, however, that these conditions can be dramatically relaxed. Provided that all distributions in the agent's choice set belong to the same linear class—a condition satisfied by most published expected utility models—arbitrary von Neumann-Morgenstern utility functions and arbitrary probability distributions can be exactly represented by the $\mu$–$\sigma$ approach.

To be more specific, let an agent's end-of-period wealth be a random variable $Y$, which is a function $y$ of the state of nature $\theta$ and some vector of choice variables $q$, and let $\mu$ and $\sigma$ be respectively the expectation and standard deviation of $Y$. Then all attainable distributions of $Y$ are said to belong to the same linear class if $y(\theta,q) = \mu(q) + \sigma(q)x(\theta)$; i.e., if they all have a common standardized form[1] $X = (Y - \mu)/\sigma$ whose properties are independent of the choice variables $q$. An example is the portfolio selection problem $y(\theta, q) = [1 + qp(\theta)]y_0$ where $1 - q$ is the proportion of initial wealth $y_0$ invested in money and $q$ is the proportion invested in a risky asset whose rate of return is $p$. Here $\mu(q) = [1 + q\mathrm{E}(P)]y_0$, $\sigma(q) = q[\mathrm{var}(P)]^{1/2}y_0$, and $x(\theta) = [p(\theta) - \mathrm{E}(P)][\mathrm{var}(P)]^{1/2}$.

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Obviously, given the properties of the linear class as defined by $x(\theta)$ and the corresponding probabilities for $\theta$, the location parameter $\mu$ and the scale parameter $\sigma$ carry all the information necessary to describe $Y$. Thus, a given von Neumann-Morgenstern function $u(y)$ must imply a unique indifference map in a $\mu$-$\sigma$ diagram for each given linear class, provided that $\mathrm{E}[u]$, $\mu$, and $\sigma$ exist.

It is true that the shape of the indifference map will, in general, depend on the properties of the linear class. However, there are some important properties of this map that hold throughout and that rehabilitate the role of the $\mu$-$\sigma$ criterion as an attractive tool for analyzing decision problems under risk. For example, when the ordinate measures $\mu$ and the abscissa $\sigma$, the indifference curves start horizontally at the ordinate, bend upwards, and are strictly convex in the case of global risk aversion ($u'' < 0$). Moreover, the indifference curve slope rises, stays constant, or declines with a vertical upward movement depending on whether the von Neumann-Morgenstern function exhibits increasing, constant, or decreasing absolute risk aversion.

While the cited results and additional properties of indifference curves were derived by Meyer and Sinn, there is one problem that has not yet been resolved and will therefore be studied here. The problem concerns the question of whether or not the indifference curve slope rises with an increase in $\sigma$, given $\mu$. To know how the indifference curve slope changes with horizontal moves in the $\mu$-$\sigma$ diagram may be useful for studying problems involving changes in the riskiness of decision alternatives such as those brought about by redistributive taxation, changed volatility in asset prices, or the purchase of insurance. In any event, the case of horizontal moves in the $\mu$-$\sigma$ diagram seems to be no less important than the thoroughly investigated case of vertical moves.

## 2. Sufficient conditions for an increased indifference curve slope

To see how the indifference curve slope changes with an increase in $\sigma$, given $\mu$, it is useful to introduce the function $s(\mu, \sigma)$ that indicates the *indifference curve slope* at a certain point in $\mu$-$\sigma$ space (Sinn, 1985). With a linear distribution class where all members have the same distribution of the *standardized random variable*

$$X \equiv \frac{Y-\mu}{\sigma} \tag{1}$$

and where it is hence possible to write expected utility as $\mathrm{E}[u(Y)] = \mathrm{E}[u(\mu + \sigma X)]$, the function $s$ can be specified as

$$s(\mu, \sigma) \equiv \left.\frac{\mathrm{d}\mu}{\mathrm{d}\sigma}\right|_{\mathrm{E}[u]} = -\mathrm{E}[Xu'(\mu + \sigma X)]/\mathrm{E}[u'(\mu + \sigma X)]. \tag{2}$$

Let

$$r(y) \equiv -u''(y)/u'(y) > 0 \tag{3}$$

be the *Pratt-Arrow measure of absolute risk aversion* and

$$g(y) \equiv r'(y)/r(y) \tag{4}$$

be a function that measures the *speed of increase in absolute risk aversion.*

As mentioned above, it is known that risk aversion ($r > 0$) implies positively sloped and convex indifference curves (Meyer, 1987, pp. 424, 425; Sinn, 1983, pp. 116, 118):

$$s > 0, \left.\frac{d^2\mu}{d\sigma^2}\right|_{E[u]} = s_\sigma + s_\mu \cdot s > 0 \qquad \text{for } \sigma > 0,$$

and it is also known that decreasing or constant absolute risk aversion implies that the indifference curve slope respectively declines or stays constant with an increase in $\mu$, given $\sigma$ (Meyer, 1987, p. 425; Sinn, 1983, p. 117):

$$s_\mu \begin{Bmatrix} < \\ = \end{Bmatrix} 0 \qquad \text{for } g \begin{Bmatrix} < \\ = \end{Bmatrix} 0 \text{ and } \sigma > 0.$$

Taken together, the properties ensure that

$$s_\sigma > 0 \qquad \text{for } g \leq 0 \text{ and } \sigma > 0. \tag{5}$$

Thus it is clear that the indifference curve slope increases with an increase in $\sigma$, given $\mu$, if absolute risk aversion is *nonincreasing.*

The way the indifference curve slope changes with an increase in $\sigma$ under *increasing* absolute risk aversion is, however, less clear. The remainder of this section addresses this problem.

To find the sign of $s_\sigma$ in the case of increasing absolute risk aversion ($g > 0$), differentiate equation (2) for $\sigma$. This gives

$$\operatorname{sgn} s_\sigma = \operatorname{sgn}\{-E[X^2u'']E[u'] + E[Xu'']E[Xu']\}. \tag{6}$$

After dividing by $E[u']$, using equations (2) and (3), factoring out $Xu''$, and substituting

$$\alpha \equiv Xr, \tag{7}$$

one obtains

$$\operatorname{sgn} s_\sigma(\mu, \sigma) = \operatorname{sgn} E[\alpha u' \cdot (X + s)].$$

By the definition of $s$ given in equation (2), $E[u' \cdot (X + s)] \equiv 0$. Moreover, $u'(\mu + \sigma x)\cdot [x + s(\mu, \sigma)]$ changes sign once from negative to positive as $x$ increases. These properties imply that

$$s_\sigma \geq 0 \qquad \text{if } \partial\alpha/\partial x \geq 0$$

and when $\sigma > 0$, as is assumed. To see this, notice that $E[(c + \alpha)u' \cdot (X + s)] = E[\alpha u' \cdot (X + s)]$ when $c$ is a constant and choose $c$ such that $c + \alpha = 0$ when $x + s = 0$. When both $(c + \alpha)$ and $(x + s)$ are increasing functions of $x$ that have identical signs for all values of $x$, all variates of their product are positive. It follows that $E[\alpha u' \cdot (X + s)] \geq 0$ and thus $s_\sigma \geq 0$ if $\partial\alpha/\partial x \geq 0$.[2]

Now, equations (7) and (3) reveal that $\partial\alpha/\partial x = r + xr'(y)\partial y/\partial x$. Together with $r > 0$ (from equation (3)), $x = (y - \mu)/\sigma$ (from equation (1)), and $g = r'/r$ (from equation (4)), this implies that the condition $\partial\alpha/\partial x \geq 0$ can be written as

$$g(y)(y - \mu) + 1 \geq 0. \tag{8}$$

By assumption, the utility function exhibits increasing absolute risk aversion; i.e., $g > 0$. Thus, equation (8) is satisfied strictly for $y \geq \mu$, and it is sufficient for $s_\sigma > 0$ that the speed of increase in absolute risk aversion satisfies the condition

$$g(y) \leq \frac{1}{\mu - y} \qquad \text{for } y < \mu. \tag{9}$$

This expression resembles one that describes the speed of increase in absolute risk aversion in the case of a quadratic utility function. The general class of quadratic utility functions is given by

$$u(y) = a + b[y - y^2/(2m)] \tag{10}$$

where $a$ is an arbitrary constant, $b$ a strictly positive constant, and $m$ the bliss level of $y$—i.e., the level implicitly defined by $u'(m) = 0$. To recognize the similarity, note that equations (3) and (4) imply in general that $g = r + u'''/u''$ and that $g = r$ in the quadratic case, since $u''' = 0$. Using this information, it can easily be shown that equation (10) implies

$$g(y) = \frac{1}{m - y} \qquad \text{(quadratic utility).}$$

It is obvious from this expression that, in the quadratic case, equation (9) is strictly satisfied if $m > \mu$. Moreover, with any given $y$, the value of $g$ in the quadratic case is a monotonically declining function of $m$. Together with the known property that $s_\sigma > 0$ for $g \leq 0$ from equation (5), this gives the following result.

**Proposition.** An increase in $\sigma$, given $\mu$, raises the indifference curve slope for a given linear class and a given utility function $u(y)$

- if $u(y)$ exhibits constant or decreasing absolute risk aversion, or
- if $u(y)$ exhibits increasing absolute risk aversion, provided that, in the range $y < \mu$, the speed of increase in absolute risk aversion, $g(y)$, does not exceed that of the "fastest" quadratic utility function compatible with positive marginal utility (i.e., the function with $m = \mu$).

To the best of the author's knowledge, no one has ever proposed or used in theoretical models a von Neumann-Morgenstern function whose absolute risk aversion increases at a speed above that characterizing quadratic functions. The proposition can therefore be taken to cover all relevant cases. It establishes that, with linear distribution classes, the indifference curves in μ–σ space become steeper with an increase in σ given μ. Increased riskiness increases the required marginal compensation for risk.

## Notes

1. In this article, functions and variates of random variables are written in lower case letters and random variables are capitalized.
2. An alternative way of proceeding is to divide equation (6) by $E[u']$ and $E[Xu'']$, assuming that $u''' > 0$, which implies that $E[Xu''] = \text{cov}(X, u'') > 0$. This gives

$$\text{sgn}\, s_\sigma = \text{sgn}\left\{-E\left[X\frac{Xu''}{E[Xu'']}\right] + E\left[X\frac{u'}{E[u']}\right]\right\}$$

or, using the definitions of $r$ and $\alpha$ from equations (3) and (7),

$$\text{sgn}\, s_\sigma = \text{sgn}\left\{-\text{cov}\left[X, \frac{-\alpha u'}{E[-\alpha u']}\right] + \text{cov}\left[X, \frac{u'}{E][u']}\right]\right\}.$$

This expression shows that $s_\sigma \geq 0$ if $\partial(-\alpha)/\partial x \leq 0$ or, equivalently, if $\partial\alpha/\partial x \geq 0$, for then the first of the two covariance terms dominates the second.

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